JOURNAL OF APPROXIMATION THEORY 72, 138–140 (1993)

Uniform Approximation and Fine Potential Theory*

P. M. GAUTHIER

Département de mathématiques et de statistique, Université de Montréal, Montréal, PQ H3C 3J7, Canada

AND

S. LADOUCEUR

Université Louis Pasteur, Strasbourg, France

Communicated by Allan Pinkus

Received March 20, 1991; accepted October 21, 1991

We give necessary and sufficient conditions for a function defined on a closed subset of $\mathbb{R}^N$ to be the uniform limit of harmonic functions.

Let F be closed subset of $\mathbb{R}^N$, $N \geqslant 2$. We denote by $\bar{H}(F)$ the closure in the topology of uniform convergence on F of the space of all harmonic functions on (neighbourhoods of) F.

We shall make use of the fine potential theory for which we refer the reader to [F1, F3, F4].

THEOREM. *Let F be a closed subset of $\mathbb{R}^N$ and u a complex-valued function on F. Then $u \in \bar{H}(F)$ if and only if*

(1) *u is continuous on F;*

(2) *u is finely harmonic on the fine interior of F.*

Proof. The case when F is compact was proved by Debiard and Gaveau [DG] (see also [BH]). For closed sets, the proof follows from the compact case and the localization theorem for harmonic functions on closed sets [GH].

Indeed, suppose that $u \in \bar{H}(F)$. Then, if K is any closed ball, the restric-

* Research supported by NSERC-Canada and FCAR-Quebec.

0021-9045/93 $5.00

tion $u|(F\cap K)$ of u to the compact set $F\cap K$ is in $\bar{H}(F\cap K)$. Thus by the Theorem of Debiard and Gaveau, u is continuous on $F\cap K$ and finely harmonic on the fine interior of $F\cap K$. Since this is true for any closed ball K, it follows that u is continuous on F and finely harmonic on the fine interior of F.

Conversely, suppose that u is continuous on F and finely harmonic on the fine interior of F. Again, let K be any closed ball. Since no point of the boundary of K lies in the fine interior of $F\cap K$, it follows that u is continuous on $F\cap K$ and finely harmonic on the fine interior of $F\cap K$. By the Debiard and Gaveau Theorem, $u|(F\cap K)\in\bar{H}(F\cap K)$. Since this is so for every closed ball K, it follows from the localization theorem for harmonic approximation on closed sets [GH, Theorem 2.3.2 and Corollary 2.3.8] that $u\in\bar{H}(F)$.

Remarks. (1) Fine potential theory is usually investigated on domains which admit nonconstant positive superharmonic functions. This would, at first, seem to exclude the plane $\mathbb{R}^2$. However, if U is any finely open set in $\mathbb{R}^2$, we may define a function to be finely harmonic on U if its restriction to the intersection of U with any ball is finely harmonic.

(2) If Ω is an open set in $\mathbb{R}^N$ and F is a subset of Ω which is closed in the relative topology of Ω, then our theorem (and its proof) still hold. The more general situation where Ω is a Riemannian manifold is currently being considered by Bagby and Blanchet [BB].

(3) An analogous result also holds for approximation by continuous subharmonic functions. For compact sets this is due to Bliedtner and Hansen [BH] (see also [F2]), while for closed sets the result is currently being written [G].

References

[BB] T. Bagby and P. Blanchet, Uniform harmonic approximation on Riemannian manifolds, *J. d'Analyse Math.*, to appear.

[BH] J. Bliedtner and W. Hansen, Simplicial cones in potential theory, I, *Invent. Math.* **29** (1975), 83–110; II, *Invent. Math.* **46** (1978), 255–275.

[DG] A. Debiard and B. Gaveau, Potentiel fin et algèbres de fonctions analytiques, I, *J. Funct. Anal.* **16** (1974), 289–304; II, *J. Funct. Anal.* **17** (1974), 296–310.

[F1] B. Fuglede, "Finely Harmonic Functions," Lecture Notes in Mathematics, Vol. 289, Springer-Verlag, New York/Berlin, 1972.

[F2] B. Fuglede, Localization in fine potential theory and uniform approximation by subharmonic functions, *J. Funct. Anal.* **49** (1982), 57–72.

[F3] B. Fuglede, Fine potential theory—A survey, *Mitt. Math. Ges. DDR* **2/3** (1986), 3–21.

[F4] B. Fuglede, Fine potential theory, *in* "Potential Theory Surveys and Problems," pp. 81–97, Lecture Notes in Mathematics, Vol. 1344, Springer-Verlag, New York/Berlin, 1988.

[G] P. M. Gauthier, Untitled manuscript, in preparation.

[GH] P. M. Gauthier and W. Hengartner, "Approximation Qualitative sur des Ensembles Non Bornés," Séminaire de Mathématiques Supérieures, Presses de l'Université de Montréal, 1982.